Asking Them "Why?" Before They Get to High School:

A Proposal for Change in Middle School Mathematics

An Honors Thesis (HONRS 499)

by

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Abstract

This project explores research conducted in the last decade over reasoning and its role in the mathematics classroom. Upper-level math teachers continually say that most students' ability to think logically and explain themselves within a mathematical context is lacking. I attribute this deficiency to the absence of opportunities for students to use reasoning and proof, even informally, during the elementary and middle school years. I analyze research carried out with case studies that supports the argument that students *can* learn to support their thinking process and communicate it to others, when given regular *opportunities* to do so.

This project also analyzes some common assessment questions given within a middle school mathematics classroom. Most of these questions come directly from *Mathematics: Applications and Concepts (Course 2)* from Glencoe/McGraw-Hill, *Middle School Math* from McDougal Littell, or the sample ISTEP+ Grade 8 mathematics questions from 2007. A few of them I created based on the experience of being asked those types of questions. I have examined 10 questions from different areas of middle school mathematics strands, considered how much they really assess *reasoning*, and provided revisions on each question to better correlate with this standard. I have also included rationales for each revision made.

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Ask the typical high school student to explain the board game *Clue*, and, assuming he has played before, he can probably do so without great difficulty. It's a familiar game within many households, taking on the fun and suspense of a "murder mystery" while simultaneously testing players' deductive thinking skills. If asked, it's likely that the student will even be able to verbally explain the reasoning behind his strategy during a particular game. Why *can't* the answer be Colonel Mustard in the kitchen with the candlestick? Now ask that same student to name the most difficult part of his high-school geometry course, and a reasonable prediction is that he'll say something along the lines of "proof." Educators routinely observe that the ability of many high school students to adequately use formal logic within a mathematics classroom is lacking. The result of this underdeveloped skill lies with previous mathematics curricula, specifically those that do not thoroughly incorporate reasoning and proof throughout their scope and sequence. A middle school mathematics curriculum that provides regular opportunities for reasoning will therefore prepare students for the rigorous components of both secondary and post-secondary mathematics courses.

The National Council for Teachers of Mathematics (2000) clearly describes reasoning as "an integral part of doing mathematics," with students in grades 6-8 "deepening their evaluations of their assertions and conjectures and using inductive and deductive reasoning to formulate mathematical arguments" (p. 262). This objective has not, however, always been emphasized in the areas of mathematics outside of traditional Euclidean geometry (Friedlander & Hershkowitz, 1997). For instance, arithmetic and algebra standards have typically focused on straightforward, drill and practice methods in order to teach the concepts (Yackel & Hanna, 2003), and Friedlander and Hershkowitz (1997) highlight that the notion of "using algebra as a tool for reasoning, explaining, and justifying is seldom addressed or emphasized, in spite of its being a direct and natural continuation of the recommended processes of generalizing" (p. 442). This lack of proof outside of geometry gives students a skewed view of the role and nature of proof in mathematics, presenting them with what Knuth (2002) refers to as "'a totally falsified picture of mathematics itself'" (p. 486).

Such an outlook can lead to an incorrect or shallow understanding of mathematical concepts. If, in algebra, emphasis is placed only on manipulating algebraic symbols and not also on arguing why such manipulations make sense (e.g. through representation), then students will emerge from an algebra class with an understanding of how to solve algebraic equations, but with a very small idea of what algebra *is*. This results in what Lannin, Barker, and Townsend (2006) call an "inability to model situations algebraically and apply the concept of variable to appropriate situations" (p. 441), a necessary skill in upper-level mathematics. The problem intensifies when students view proof as rigid and pointless, often the result of teachers placing too much stress on the technical details rather than educational value of proof (Knuth, 2002). In this case, students do not see proof as a *tool*; rather, they see it as an obstacle.

This negative connotation with reasoning and proof can also be a result of its level of difficulty for some students. Research indicates that most students struggle to comprehend a formal, deductive argument (Healy & Hoyles, 2000), which can be caused by lack of conceptual understanding, inexperience with mathematical language and notation, and uncertainty of how to get started on a proof (Yackel & Hanna, 2003). Yackel and Hanna (2003) also note that this difficulty can arise because students "'begin their upper-level mathematics courses having written proofs only in high school geometry and having seen no general perspective of proof or methods of proof'" (p. 232). Similarly, Knuth (2002) writes "because of such limited experiences with proof, many secondary school mathematics students have found the study of proof difficult," (p. 486), and he also notes that many students do not understand what constitutes proof.

These difficulties of secondary students indicate that not only should proof be integrated within areas of secondary mathematics besides just geometry, but also that a greater emphasis should be placed on general reasoning, particularly in the elementary and middle grades. One basis for this lack of reasoning in *middle school* mathematics, particularly algebra, is that, according to Friedlander and Hershkowitz (1997), "students have difficulty making connections between algebra and reasoning," and they are under the impression "that algebra does not have the potential to prove anything" (Friedlander p. 442). Therefore, teachers must offer students *opportunities* that will demonstrate one branch's connection to other mathematical topics (i.e. algebra to geometry), as well as that subject's usefulness in analyzing, reasoning, and justifying (Friedlander & Hershkowitz, 1997). Finding such worthwhile tasks can be difficult, and revising textbook and standardized assessment exercises is often necessary in order to achieve this goal.

Even if teachers do accept that most traditional, textbook-based curricula do not offer consistent engagement in reasoning activities, how will they begin to revise such strategies to better align with this process? Simply put, what should reasoning look like within a mathematics classroom? To start, Healy and Hoyles (2000) note that, in terms of formal proof, students are more likely to gain understanding when the emphasis is on communication rather than rigor. Can the student first give, follow, and understand an informal argument? If not, then the focus should not yet be on the rigid mechanics. Yackel and Hanna (2003) also support this notion, saying that "the functions of proof that may have the most promise for mathematics education are those of explanation and communication" and that "proof is much more than a sequence of logical steps; it is also a sequence of ideas and insights" (p. 228). They describe the mistake that most educators make in their approach of reasoning: treating proof as being purely instrumental mathematics, rather than instrumental and relational mathematics. That's like only being able to follow step-by-step directions, rather than learning to read and interpret a map (Yackel & Hanna, 2003).

Instead, emphasizing to students that the aim of mathematical reasoning is not simply to produce a solution, but to produce knowledge, is vital in developing their skills (Yackel & Hanna, 2003). Friedlander and Hershkowitz (1997) outline a four-stage sequence for algebra tasks that encompasses such a philosophy: start with a "construction"—an "emerging pattern" of specific examples—that is to be "discovered, applied, investigated, and justified;" move toward a "working generalization"—possibly nonverbal—by "producing additional examples or solving 'reversal' tasks;" work toward an "explicit generalization" by requiring verbalization of observations; and finally work toward a "justification" by convincing others that such observations and conclusions make sense (p. 444). Bremigan (2004) goes further by specifically suggesting that beginning algebra teachers use consecutive integers as the emerging pattern, "since the choice of notation is relatively easy to understand and since only basic algebraic skills are required to produce the proofs" (p. 97). Yackel and Hanna (2003) call for the use of "preformal" proofs, suggesting that students develop a "chain" of informal, yet correct, conclusions "starting from valid, nonformal premises" (p. 233), and Knuth (2002) notes the importance of using counterexamples as a sophisticated method of proof (p. 489).

Lannin, Barker, and Townsend (2006) also highlight four stages, but instead of outlining a straightforward process, they instead focus on four categories of student justifications. The first is no justification or a procedural justification, which can occur when students do not explain how they derive a generalization or why such a generalization makes sense. The next is an empirical justification, which expands on the first category by at least providing a test of several cases, adding some support to the validity of a generalization. More refined than this case is that of generic examples, which takes a specific case but uses it to "communicate generality across cases." Finally, there is the case of deductive justification, which "provides a general argument that clearly explains why the rule applies to all cases of the situation." Given each of these categories, teachers can more easily determine what signifies a valid argument (p. 440).

Even with these guidelines for what reasoning should look like, the act of planning and applying such approaches still presents a challenge to educators (Knuth, 2002). Knuth (2002) sums up these strategies best in an excerpt from Kenneth Ross, saying "'the emphasis of proof should be more on its educational value than on formal correctness. Time need not be wasted on the technical details of proofs, or even entire proofs, that do not lead to understanding or insight'" (p. 487). In algebraic thinking, specifically, the focus should move beyond the elementary school emphasis on exact calculations, and instead to a practice of constructing generalizations (Lannin, Barker, & Townsend, 2006).

Specific research studies show the positive effects of such practices. One study documented by Healy and Hoyles (2000) uses a questionnaire to determine students' views of the notion of proof from a variety of standpoints:

> First, students were asked to provide written descriptions about proof and what they thought it was. Second, students were presented with a mathematical conjecture and a range of different types of arguments in support of it. They were then asked to make a selection from these arguments with two criteria—which argument would be nearest to their own approach and which they believed would receive the best mark from the teacher. Third, students' assessments of these arguments in terms of their validity or explanatory power were elicited. (p. 184)

What makes this study distinct is the "strong statistical difference" between what students said would be their own approach when arguing the conjecture and what they chose to receive the best mark. In fact, Healy and Hoyles note that "the arguments that were the *most* popular for the students' own approaches turned out to be the *least* popular when the students chose an approach to receive the best mark, and vice versa" (p. 187). When teachers were asked to complete this same questionnaire based on their own preferred approaches and what they believed their students would choose for the best mark, both answer choices tended to be the same. Breaking down both sets of data, it seems that students believed teachers would be satisfied with *any* argument that at least contains some "algebra," whereas teachers expect to see some logic presented in the argument also (Healy & Hoyles, 2000). Further analysis concludes that even though the students typically used empirical arguments when constructing their own proofs, they recognized that such arguments would not receive high marks. Also, the majority of students were aware that a valid proof must be general, even though they might differ with teachers in believing that "complicated algebraic expressions" would still receive high marks (Healy & Hoyles, 2000). Overall, Healy and Hoyles (2000) conclude that "students were significantly better at choosing correct mathematical proofs than at constructing them" (p. 188).

Yackel and Hanna (2003) focus on another study conducted by Maher and Martino in 1996. In this particular study, one child's course of development in terms of mathematical reasoning is tracked over a 5-year period. This child, who they name Stephanie, began the study in first grade and continued through fifth grade, working on combinatorics tasks throughout the five years. Yackel and Hanna (2003) give particular interest to the Towers Problem, "in which the task is to figure out how many different towers four (or five) cubes tall can be made selecting from red and blue cubes" (p. 231). Stephanie works on this problem at various times in the last three years of the study, and her method of justification progresses as time goes on. She is initially recorded as using "trial-and-error and guess-and-check strategies to create new towers and search for duplicates...[but] by the middle of her fourth-grade year, [she] introduced an indirect method of proof to account for all possibilities" (p. 231). She later explores using a proof by cases during an interview setting, going on to present this proof to her classmates (Yackel & Hanna, 2003).

Maher and Martino emphasize that the learning environment in this study provided multiple opportunities for children to explore ideas, often times by allowing for, according to Yackel and Hanna (2003), a "flexibility of content and extended periods of time for exploration" (p. 231). They note that Stephanie's mathematical growth throughout the five years suggests that notions of formal proof can "develop naturally out of an emphasis on justification and explanation" (p. 231), a notion that elementary and middle-school educators should take back to their classrooms. Yackel and Hanna (2003) note that this and similar studies show that students do not need to wait until high school to begin justifying their and others' work; they can engage in mathematical reasoning as early as elementary school. Of course, not all students develop their sophistication in reasoning at the same pace (Yackel & Hanna, 2003), nor should they be expected to. Friedlander and Hershkowitz (1997) highlight a study on generalizing and justifying patterns in which students in different stages of algebraic development (from pre-algebra to second-year) and of various levels of mathematical ability work in different settings and "quite satisfactorily" perform initial stage activities. They note that "the fact that *all* students could get started and perform according to their level...can in itself be viewed as an advantage and a good reason to use this type of activity in the process of learning algebra" (p. 444-45). They go on to say that most students observed in this study were in fact able to go on to the stages of generalization and justification, and from this study they indicate that "problem situations based on the generalization and justification of patterns help emphasize a frequently neglected aspect of algebra—its ability to promote mathematical reasoning and an understanding of the nature of mathematical proofs" (p. 446-47). Also, it seems that no matter the case study, one fact is clear: there exists a strong correlation between the nature of mathematical explanations and justifications and actual mathematical learning (Yackel & Hanna, 2003).

So taking in these research studies and the reasoning guidelines provided, what is the overall outcome of choosing whether or not to regularly implement proof into the mathematics classroom? Some positive outcomes of implementation have already been listed, but to add further emphasis, Bremigan (2004) highlights the explaining of observed patterns (particularly in algebra) and the development of a deeper mathematical understanding. She also notes that "opportunities abound for students to form conjectures and develop proofs in other mathematics content areas" (p. 100) other than geometry and measurement, and associates the seizing of such opportunities with deeper mathematical understanding. Finally, Lannin, Barker, and Townsend (2006) warn that, "if discussion about what constitutes a valid justification does not occur, students often rely on the superficial aspects of an argument, such as the use of formal mathematical symbols, over the mathematical reasoning that underlies the argument" (p. 440). We see this result in the noted studies from Healy and Hoyles (2000) of students who believed that teachers would be satisfied with *any* argument, so long as it contained *some* algebra.

In conclusion, we see that the positive outcomes of implementing reasoning and proof, *especially* in the elementary and middle school grades, can be worth the effort of revising select textbook and assessment exercises, especially considering the potential long-term effects on students. Proof should *not* be a foreign concept to the beginning high-school geometry student, and given a teacher willing to provide opportunities for reasoning in these earlier grades, it does not need to be. A middle school mathematics curriculum developed with these guidelines in mind can properly prepare students for more rigorous mathematics courses, and maybe even ease a student's apprehension of entering a post-secondary mathematics-related field. After all, if students can play "detective" in a board game, then obviously the ability to think logically is present; teachers must simply take the initiative to probe that type of thinking inside the classroom.

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Data and Statistics:

Common Item:

Find the mean, median, and mode for the following set of data:

2, 7, 2, 7, 13, 7, 11, 9, 6, 5, 8

(Larson, Boswell, Kanold, & Stiff, 2004).

Revised Item:

Given the following set of data:

2, 7, 2, 7, 13, 7, 11, 9, 6, 5, 8

(a) Calculate the mean, median, and mode for the data.
(b) In your own words, explain why your calculated mean makes sense.
(c) Suppose the number 13 were changed to 20. Explain how this change would affect the mean, median, and mode, and justify your response.
(d) Suppose one of the 7's were changed to a 2. Explain how this change would affect the mean, median, and mode, and justify your response.

Rationale:

NCTM Standard:

Data Analysis and Probability

Find, use, and interpret measures of center and spread, including mean and interquartile range.

Indiana Standard:

7.6.3 Describe how additional data, particularly outliers, added to a data set may affect the mean, median, and mode.

The concept of mean, median, and mode is too often taught on a solely procedural level, with any conceptual instruction usually in the form of technical definitions. Sampled textbooks include several exercises asking students to "find the mean" and "find the median, mode and range," while only 2-3 exercises are dedicated to "choosing the best average." This last objective requires students to analyze the data rather than just perform a calculation, and I have extended the exercise above in a similar manner. Part (a) lets a teacher examine a student's calculations of mean, median, and mode, while part (b) allows for analysis of a student's conception of "mean." Parts (c) and (d) require students to not only perform a calculation using entries in a data set, but also to explain how changing these entries affects the measures of central tendency.

Linear Equations and Functions:

Common Item:

Solve the equation $22 = 5.5x$

Revised Item:

Suppose you have the following table. Fill in the missing entries.

x	y
1	5.5
	22
7	
	42.63

Explain how you determined the missing entries. Then, express the relationship between x and y as an algebraic equation.

Rationale:

NCTM Standard:

Algebra

Relate and compare different forms of representation for a relationship.

Indiana Standard:

6.3.1 Write and solve one-step linear equations and inequalities in one variable and check the answers.

The concept of function is a foundation of upper-level math courses, and understanding the meaning and behavior of functions requires a firm grasp of mathematical relationships. Too often, the equation above is viewed only in symbolic notation, and thus the steps to solve the equation are described in the same context: write original equation, divide each side by 5.5, and simplify. I revised this exercise to provide students the opportunity to view this equation as a **relationship**, and not as mere algebraic symbols. They can then formulate and justify the same equation using the relationship they discovered within the table.

<u>Algebraic Thinking:</u>

Common Item:

> Irene left both her dog and cat at a kennel for 3 nights. The kennel charges $8 per night for the dog and $5 per night for the cat. Find Irene's bill.

(Bailey et al., 2004)

Revised Item:

> Irene needs to take both her dog and cat to a veterinarian for surgery. There are two vets in her town. Vet A charges $50 for the surgery, plus $8 per night for the dog and $5 per night for the cat. Vet B requires $60 for the surgery, but charges $6 per night for the dog and $3 per night for the cat. Explain how you would determine which vet Irene should choose if both her pets need to stay for two nights after surgery. Should she choose this same vet if her pets need to stay for three nights? Explain.

Rationale:

<u>NCTM Standard:</u>

Algebra

> Model and solve contextualized problems using various representations, such as graphs, tables, and equations.

<u>Indiana Standard:</u>

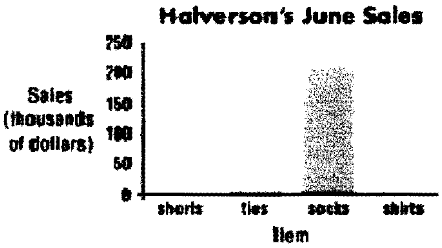
7.3.1 Use variables and appropriate operations to write an expression, a formula, an equation, or an inequality that represents a verbal description.

A misconception among naïve educators is that a word problem in mathematics automatically constitutes problem solving and reasoning. While the problem above does require some algebraic thinking, reasoning is not addressed strongly, given the problem is merely an arithmetic computation in the context of a veterinary bill. My revised word problem addresses the same computational skills, but also introduces the case of a nonzero constant in a linear equation. Students must not only perform the computations, but also realize that the correct answer will depend upon the input value for number of nights spent after surgery. Reasoning methods for this problem opens the door for students to utilize several means of representation, including writing algebraic equations, graphing two lines and determining the intersection, or making a table.

<u>Interpreting Graphs:</u>

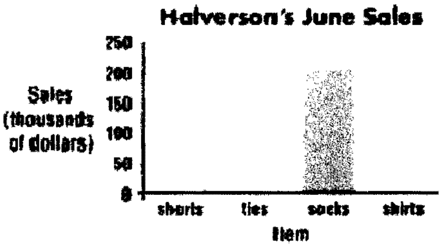
Common Item:

Predict which two items' sales will probably combine to bring in $350,000 next month.



Revised Item:

The graph below describes one month's sales by item for a particular company.



(a) Predict which two items' sales will probably combine to bring in $250,000 next month.
(b) Could part (a) have more than one possible answer? Explain.
(c) In July, Halverson's sales goal is $400,000, but the company must also choose an item to stop selling. Which item(s) could Halverson choose to stop selling, yet still meet their sales goal? Explain your reasoning.

Rationale:

<u>NCTM Standard:</u>

Data Analysis and Probability

Select, create, and use appropriate graphical representations of data, including histograms, box Plots, and scatterplots.

<u>Indiana Standard:</u>

7.6.1 Analyze, interpret, and display data in appropriate bar, line, and circle graphs and stem-and-leaf plots and justify the choice of display.

While the exercise above focuses solely on interpreting a bar graph, my expanded version also asks students to consider multiple answers to the same question and to make decisions based on given data. This allows for justification in that students can explain how they interpret the graph and also how they divide a sum into its different parts. The answers alone are not difficult to determine, but requiring a written explanation of the reasoning provides students with a wide perspective of the situation.

Geometry:

Common Item:

Find the area of a triangle whose base is 4 feet and whose height is 17 feet.

Revised Item:

Explain why the area of a triangle is ½*bh*.

[Hint: Draw a picture.]

Rationale:

NCTM Standard:

Geometry

Understand relationships among the angles, side lengths, perimeters, areas, and volumes of similar objects.

Indiana Standard:

8.4.1 Identify and describe basic properties of geometric shapes: altitudes, diagonals, angle bisectors, perpendicular bisectors, central angles, radii, diameters, and chords of circles.

In traditional secondary math curriculums, great devotion is paid to deductive reasoning and two-column proofs in geometry. However, experts state that most college-level students have difficulty following formal proofs because they only write proofs during the year they take geometry. Balacheff suggests that emphasis should be placed on students' reasoning rather than the mere written form of proof (Yackel & Hanna, 2003), and I have revised an exercise addressing the application of the area formula for a triangle. The ability to plug values into a formula does not indicate an understanding of geometry, but the derivation of a formula from a previously known relationship can. The new exercise allows students to explain the area formula, not just recall it.

Probability:

Common Item:

You randomly choose a marble from a collection of 6 red, 4 blue, and 2 green. Find the probability of choosing a red marble. A blue. A green.

Revised Item:

You have two bags of marbles. Bag #1 contains 12 marbles, each being either red, blue, or green. Bag #2 also contains 12 marbles, but with a different assortment of red, blue, and green.

(a) Can you describe a possible combination of your first 3 draws for Bag #1? Why or why not?
(b) Suppose you are only allowed to make 3 draws from each bag. After this you must put the 3 marbles back in their bag and redraw. Describe a method you can use to predict how many of each color marble are in each bag. Be sure to explain your reasoning behind this method.
(c) Your teacher will give you a prize if, in your first 4 draws from a single bag, you draw exactly 2 green marbles. What would be the optimal assortment of green marbles in this entire bag, so that you have the best chance at winning the prize? Explain your reasoning.

Rationale:

NCTM Standard:
Data Analysis and Probability

Formulate questions, design studies, and collect data about a characteristic shared by two populations or different characteristics within one population.

Indiana Standard:
8.6.1 Identify claims based on statistical data and, in simple cases, evaluate the reasonableness of the claims. Design a study to investigate the claims.

Jones, Thornton, Langrall, and Tarr (1999) describe probabilistic reasoning as having "a special place within the broader picture of mathematical reasoning because it involves reasoning that is associated with a context of uncertainty" (p. 155). As such, I've revised a common probability exercise focusing only a *theoretical* probability and have instead focused on *experimental* probability. Such a change brings students back to the main idea behind probability: its use is to **predict** when a situation is uncertain. This prediction may be completely true or completely false, but using experimental methods, students have the opportunity in part (b) to describe how to make a good guess concerning the assortment of marbles in each bag. In part (c) they use the concept of sample spaces and theoretical probability to determine that if 50% of the marbles in the sample must be green, then 50% of the marbles in the population being green would be most optimal.

<u>Measurement:</u>

Common Item:

Last Saturday, Rachel shelled walnuts. She was paid $5.00 for the day, plus an additional $.10 for each cup of walnuts she shelled. If Rachel earned a total of $17.00, how many quarts of walnuts did Rachel shell?

(*ISTEP+ publications*)

Revised Item:

Last Saturday, Rachel shelled walnuts. She was paid $5.00 for the day, plus an additional $.50 for every 4 cups of walnuts she shelled. Rachel earned a total of $17.00.

(a) If Rachel only shelled 1 cup of walnuts, how much money would she earn for the day?
(b) What measuring unit would you choose to describe the total amount of walnuts that Rachel actually shelled? Explain your reasoning.
(c) Calculate the amount of walnuts that Rachel shelled, using the unit you chose in part (a).

Rationale:

<u>NCTM Standard:</u>
Measurement

Understand relationships among units and convert from one unit to another within the same system.

<u>Indiana Standard:</u>
6.5.1 Select and apply appropriate standard units and tools to measure length, area, volume, weight, time, temperature, and the size of angles.

Whether or not they realize it, students will be often be required to use measuring skills outside of the classroom. They will need to choose appropriate units of measurement when deciding how to arrange furniture in a room, how much of a particular ingredient should be used in a recipe, and how much material they will need when taking on a home improvement project. They will also need to convert measurements when communicating this information to others. This problem requires students to first think conceptually about appropriate units **before** using a mathematical process to find the answer.

Number Patterns:

Common Item:

Find the next term in the sequence 83, 92, 101, 110, ...

Revised Item:

Consider the number sequence 83, 92, 101, 110, ...

(a) What is the next term in this sequence? Explain how you found this.
(b) What is the 100^{th} term in this sequence? Explain how you found this and why your answer is correct.
(c) Using the same pattern of change as in the sequence above, suppose you have a new sequence whose first term is 75. Find the 56^{th} term in this new sequence, and explain why your answer is correct.

Rationale:

NCTM Standard:

Algebra

Represent, analyze, and generalize a variety of patterns with tables, graphs, words, and, when possible, symbolic rules.

Indiana Standard:

6.7.1 Analyze problems by indentifying relationships, telling relevant from irrelevant information, identifying missing information, sequencing and prioritizing information, and observing patterns.

Recognizing patterns and relationships among numbers is a key component to understanding functions, and this skill can be practiced through the use of basic number patterns. Experts state that using consecutive integers is a suitable way of allowing pre-algebra students to form, test, and prove conjectures about patterns (Bremigan, 2004). As a result, I've chosen an exercise that asks students to identify a pattern and apply it to get the correct answer. My revision requires these same steps, but it also allows students to see that a "pattern" goes on indefinitely, and that understanding the relationship among the first terms in a pattern can allow them to find **any** term thereafter. They can use informal reasoning in parts (b) and (c) to both explain and justify their thought processes.

Percents:

Common Item:

If a jacket is on sale for 30% off the regular price of $65.49, how much is the discount?

Revised Item:

Suppose you find a jacket that is on sale for 30% off the regular price of $65.49. The next day you see the same jacket at a different store. Its regular price is $60.24 and it is marked at 20% off. Which store would you buy the jacket from? Explain why your answer makes sense.

Rationale:

NCTM Standard:

Number and Operations

Work flexibly with fractions, decimals, and percents to solve problems.

Indiana Standard:

6.2.8 Calculate given percentages of quantities and solve problems involving discounts at sales, interest earned, and tips.

Percents are used regularly in the real world, and I have taken the common task of finding a percentage and expanded it to include a "what would you do?" type situation. Not only must the student find the answer to a calculation, but she must also make a decision and provide **mathematical justification** for her decision. This type of task is described by researchers as being an "innovative" way to improve mathematical reasoning, specifically critical- and creative-thinking skills (Krulik & Rudnick, 1999).

Exponents:

Common Item:

Evaluate: $9^3 \cdot 9^4$

Revised Item:

Show why each law of exponents listed below is always true. If you see a statement that is not a law of exponents, explain why it is not always true.

(a) $a^3 - a^2 = a^1$
(b) $b^2 \cdot b^2 = b^4$
(c) $m^3 \cdot m^3 = m^9$
(d) $n^5 / n^3 = n^2$

Revision:

NCTM Standard:

Number and Operations

Develop an understanding of large numbers and recognize and appropriately use exponential, scientific, and calculator notation.

Indiana Standard:

8.1.5 Use the laws of exponents for integer exponents.

Too often students memorize the rule "same base, add [or subtract] exponents" when dealing with laws of exponents, but they do not receive the necessary time to explore why this rule makes sense. To promote its derivation, I've expanded upon a simple calculation that applies the rule for multiplying exponents with the same base. I've replaced the base 9 with some arbitrary value in each of the statements listed, so that students must determine if the statement is always true or not always true. Once they determine this, they must prove why that answer makes sense. This can be done either by giving a counterexample or by replacing exponential notation with repeated multiplication. Such an exercise allows students to both enhance their reasoning abilities and develop their understanding of laws of exponents.